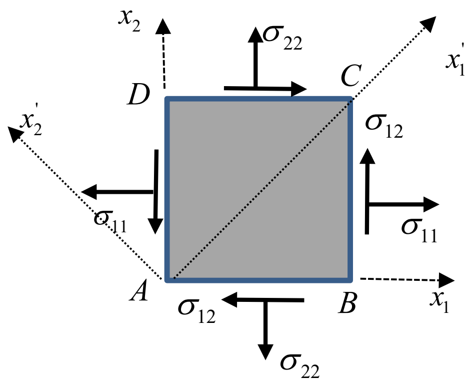
Exercise 1: A 50 mm thin square aluminum plate (Young's modulus, $E = 70GPa$ Poisson's ration v=0.3) is subjected to the stresses shown in the figure below. Calculate the change in length, of the diagonal *BD* in two ways:

(a) Determine the strains with respect to x_1, x_2 and employ the strain transformation equation.

(b) Determine the stresses with respect to $x_1^{'}, x_2^{'}$ and use the Hook's law.



$\sigma_{11} = 20\text{MPa}$
$\sigma_{22} = 10\text{MPa}$
$\sigma_{12} = \sigma_{21} = 5\text{MPa}$

Solution:

1: From the given stresses and geometry (thin plate), we have plane stress a plane stress state. Thus,

$$\varepsilon_{11} = \frac{1}{E}\left(\sigma_{11} - v\sigma_{22}\right) = 17/E$$

$$\varepsilon_{22} = \frac{1}{E}\left(\sigma_{22} - v\sigma_{11}\right) = 4/E$$

$$\varepsilon_{12} = \frac{1+v}{E}\sigma_{12} = 6.5/E$$

We use (for $\theta = 45^o$ and BD = 0.05 $\sqrt{2}$ m)

$$\varepsilon_N = \varepsilon_{BD} = \varepsilon_{11}\sin^2\theta + \varepsilon_{22}\cos^2\theta - 2\varepsilon_{12}\sin\theta\cos\theta$$
$$\Rightarrow \varepsilon_{BD} = 4/E \Rightarrow \Delta_{BD} = \varepsilon_{BD}BD = 0.283/E\ m$$

2: Apply equations

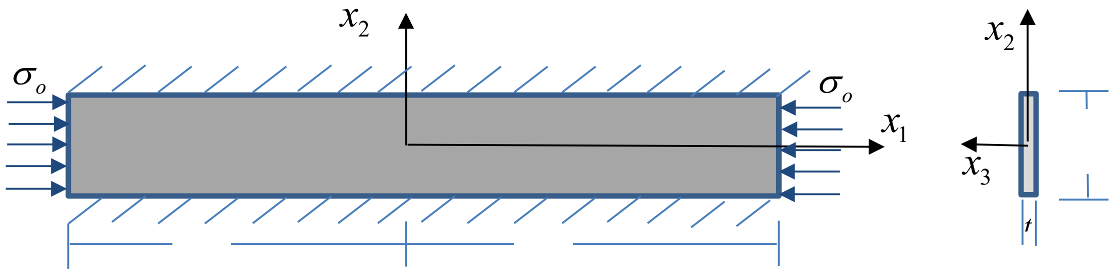
$$\sigma_{11}^{'} = \frac{\sigma_{11}+\sigma_{22}}{2} + \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta + \sigma_{12}\sin 2\theta = 20MPa$$

$$\sigma_{22}^{'} = \frac{\sigma_{11}+\sigma_{22}}{2} - \frac{\sigma_{11}-\sigma_{22}}{2}\cos 2\theta - \sigma_{12}\sin 2\theta = 10MPa$$

Then $\varepsilon_{22}^{'} = \frac{1}{E}\left(\sigma_{22}^{'} - v\sigma_{11}^{'}\right) = 4/E \Rightarrow \Delta_{BD} = \varepsilon_{BD}^{'}BD = 0.283/E\ m$

Exercise 2: The plate shown in the figure is subjected to loads that produce the uniform stress at the two ends. The long edges are placed between two rigid walls. Show that the following given displacements are correct.

$$u_1 = -\frac{1-v^2}{E}\sigma_o x_1, \qquad u_2 = 0, \qquad u_3 = \frac{v(1+v)}{E}\sigma_o x_3$$



Solution:

Stresses:

The plate is thin and no normal load is acting on it: $\Rightarrow \sigma_{33} = \sigma_{13} = \sigma_{23} = 0$.

It also confined along the edge $x_2 = \pm h$: $\Rightarrow \varepsilon_{22} = 0$

The plate is subjected to the following stresses:

$\sigma_{11} = -\sigma_0$, $\sigma_{22} = c$, $\sigma_{12} = \sigma_{21} = 0$ (c can be easily calculated by using the boundary conditions).

With these stresses, the equilibrium equations are is satisfied.

Strains:

$$\varepsilon_{11} = \frac{1}{E}(\sigma_{11} - v\sigma_{22}), \quad \varepsilon_{22} = \frac{1}{E}(\sigma_{22} - v\sigma_{11}), \quad \varepsilon_{33} = -\frac{v}{E}(\sigma_{11} + \sigma_{22}), \quad \varepsilon_{12} = \varepsilon_{21} = 0$$

$$\varepsilon_{22} = 0 \Rightarrow \sigma_{22} = v\sigma_{11} = -v\sigma_0, \quad \varepsilon_{33} = -\frac{v}{E}\left[-\sigma_0 - v\sigma_0\right] = \frac{v}{E}(1+v)\sigma_0, \quad \varepsilon_{11} = \frac{1}{E}(-\sigma_0 + v^2\sigma_0) = -\frac{\sigma_0}{E}(1-v^2)$$

Displacements:

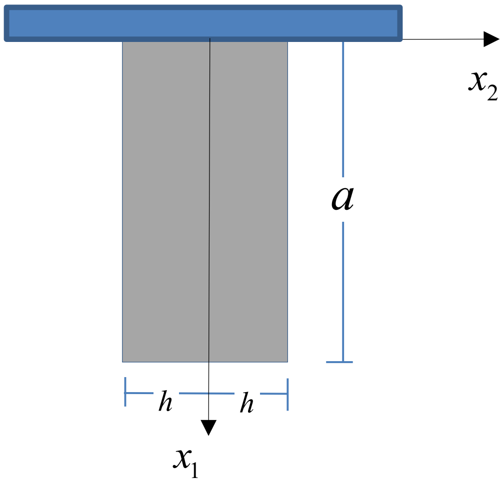
$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} \quad \Rightarrow u_1 = \int \varepsilon_{11} dx_1 = -\frac{\sigma_0}{E}(1-v^2)\int dx_1 + C_1 = -\frac{\sigma_0}{E}(1-v^2)x_1 + C_1$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial x_3} \quad \Rightarrow u_3 = \int \varepsilon_{33} dx_3 = \frac{v}{E}(1+v)\sigma_0 x_3 + C_3, \quad \varepsilon_{22} = 0 \quad \Rightarrow u_2 = 0$$

Displacement boundary conditions:

$$u_1 = 0 \quad at \quad x_1 = 0 \quad \Rightarrow \quad C_1 = 0, \quad u_3 = 0 \quad at \quad x_3 = 0 \quad \Rightarrow \quad C_3 = 0$$

Exercise 3: A thin prismatic bar of specific weight γ and constant cross section hangs vertically (see Figure). Under the effect of its own weight, the displacement field is



$$u_1(x_1,x_2) = \frac{\gamma}{2E}\left(2x_1 a - x_1^2 - \nu x_2^2\right)$$

$$u_2(x_1,x_2) = -\frac{\nu\gamma}{E}\left(a - x_1\right)x_2$$

Calculate the strain and stress components in the bar and check the boundary conditions if they are verified. The elastic properties of the bar (E,ν) are known and displacement and stress along the normal axis to the bar are neglected.

Solution:

Strains

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1} = \frac{\gamma}{2E}\left[2a - 2x_1\right] = \frac{\gamma}{E}\left[a - x_1\right]$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial x_2} = -\frac{\nu\gamma}{E}\left[a - x_1\right]$$

$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \frac{1}{2}\left[-\frac{\gamma\nu x_2}{E} + \frac{\gamma\nu x_2}{E}\right] = 0$$

Stress (plane stress)

$$\sigma_{11} = \frac{E}{1-\nu^2}\left[\varepsilon_{11} + \nu\varepsilon_{22}\right] = \frac{E}{1-\nu^2}\left[\frac{\gamma}{E}(a - x_1) - \frac{\nu^2\gamma}{E}(a - x_1)\right] = \gamma(a - x_1)$$

$$\sigma_{22} = \frac{E}{1-\nu^2}\left[\varepsilon_{22} + \nu\varepsilon_{11}\right] = \frac{E}{1-\nu^2}\left[-\frac{\nu\gamma}{E}(a - x_1) + \frac{\nu\gamma}{E}(a - x_1)\right] = 0$$

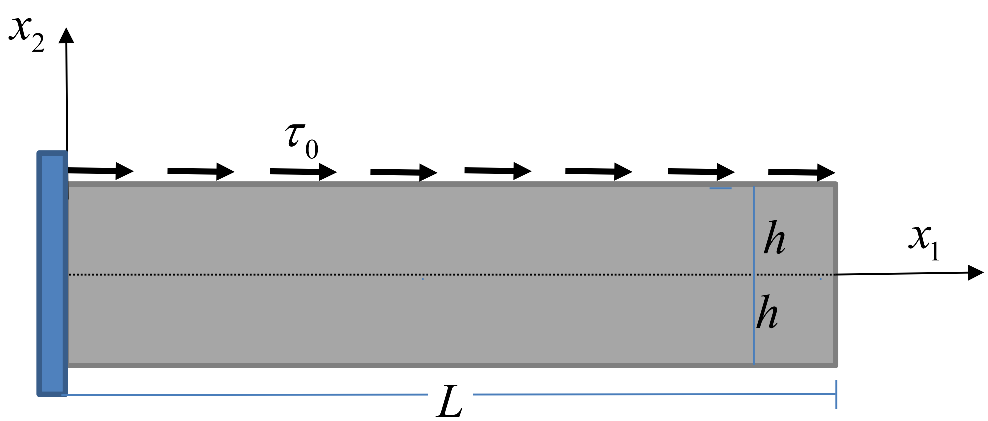
$$\sigma_{12} = \frac{E}{1+\nu}\varepsilon_{12} = 0$$

Boundary conditions

$$@\, x_1 = a \Rightarrow \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 0$$

$$@x_2 = \pm 2h \Rightarrow \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} = \begin{pmatrix} \gamma(x_1 - a) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \pm 1 \\ 0 \end{pmatrix} = 0$$

Exercise 4: The thin cantilever beam is subjected to a uniform shear stress along the entire upper surface as shown in the Figure.



Determine if the following Airy stress function is appropriate for this problem.

$$\Phi(x_1, x_2) = \frac{1}{4}\tau_0\left(x_1x_2 - \frac{x_1x_2^2}{h} - \frac{x_1x_2^3}{h^2} + \frac{Lx_2^2}{h} + \frac{Lx_2^3}{h^2}\right) \tag{a}$$

Solution

Biharmonic equation

$$\frac{\partial \Phi}{\partial x_1} = \frac{1}{4}\tau_0\left(x_2 - \frac{x_2^2}{h} - \frac{x_2^3}{h}\right)$$

$$\frac{\partial^2 \Phi}{\partial x_1^2} = 0 \quad \Rightarrow \quad \frac{\partial^4 \Phi}{\partial x_1^4} = 0 \tag{b}$$

$$\frac{\partial \Phi}{\partial x_2} = \frac{1}{4}\tau_0\left(x_1 - \frac{2x_1x_2}{h} - \frac{3x_1x_2^2}{h^2} + \frac{2Lx_2}{h} + \frac{3Lx_2^2}{h^2}\right)$$

$$\frac{\partial^2 \Phi}{\partial x_2^2} = \frac{1}{4}\tau_0\left(-\frac{2x_1}{h} - \frac{6x_1x_2}{h^2} + \frac{2L}{h} + \frac{6Lx_2}{h^2}\right)$$

$$\frac{\partial^3 \Phi}{\partial x_2^3} = \frac{1}{4}\tau_0\left(-\frac{6x_1}{h^2} + \frac{6L}{h^2}\right)$$

$$\frac{\partial^4 \Phi}{\partial x_2^4} = 0 \qquad \text{(c)}$$

$$\frac{\partial^3 \Phi}{\partial x_2^2 \partial x_1} = \frac{1}{4}\tau_0\left(-\frac{2}{h} - \frac{6x_2}{h}\right) \Rightarrow \frac{\partial^4 \Phi}{\partial x_2^2 \partial x_1^2} = 0 \qquad \text{(d)}$$

From (a), (b) and (c) we see that $\nabla^4 \Phi = 0$.

<u>Stress components</u>

$$\sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} = 0, \sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} = \frac{1}{4}\tau_0\left(-\frac{2x_1}{h} - \frac{6x_1 x_2}{h^2} + \frac{2L}{h} + \frac{6Lx_2}{h^2}\right),\ \sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} = -\frac{1}{4}\tau_0\left(1 - \frac{2x_2}{h} - \frac{3x_2^2}{h^2}\right)$$

Boundary conditions

$$x_2 = h \quad \Rightarrow \quad \sigma_{22} = 0, \quad \sigma_{12} = \tau_0$$

$$x_2 = -h \quad \Rightarrow \quad \sigma_{22} = \sigma_{12} = 0$$

$$x_1 = L \quad \Rightarrow \sigma_{11} = 0, \quad \sigma_{21} = -\frac{1}{4}\tau_0\left(1 - \frac{2x_2}{h} - \frac{3x_2^2}{h^2}\right)$$

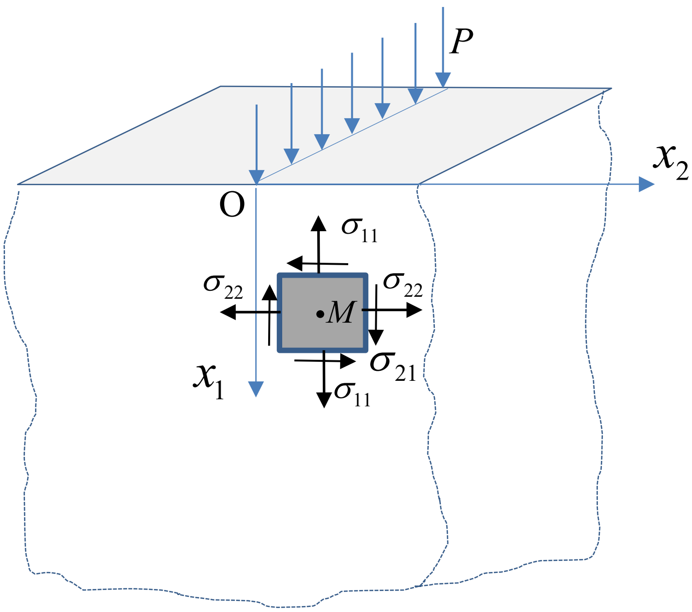
The results show that the boundary conditions are satisfied at $x_2 = \pm h$ but not at $x_1 = L$. (The surface $x_1 = L$ is free of stress while the analysis results in a non-zero stress). Thus, the Airy stress function is not appropriate.

Exercise 5: For the line load on a semi-infinite body (see Figure below)

1: Show that the stress function,

$$\Phi(x_1,x_2) = -\frac{P}{\pi}x_2\tan^{-1}\left(\frac{x_2}{x_1}\right)$$

results in the following stress field at a point *M* in the body.



$$\sigma_{11} = -\frac{2P}{\pi}\frac{x_1^3}{\left(x_1^2+x_2^2\right)^2},$$

$$\sigma_{22} = -\frac{2P}{\pi}\frac{x_1x_2^2}{\left(x_1^2+x_2^2\right)^2},$$

$$\sigma_{12} = -\frac{2P}{\pi}\frac{x_2x_1^2}{\left(x_1^2+x_2^2\right)^2}$$

2: on a plane at a distance $x_1 = a$ consider the vertical sum of forces and show that equilibrium is satisfied (Hint: consider the vertical equilibrium of a stripe of unit thickness).

Note that: $\int_{-\infty}^{+\infty}\frac{a^3}{\left(a^3+x_2^2\right)^2}dx_2 = \frac{\pi}{2}$

Solution:

1: Note that : $\frac{d}{dx}\tan^{-1}\left(f(x)\right) = \frac{df(x)}{dx}\frac{1}{1+\left(f(x)\right)^2}$

$$\frac{\partial\Phi}{\partial x_2} = -\frac{P}{\pi}\left(\tan^{-1}\left(\frac{x_2}{x_1}\right)+\frac{x_1x_2}{\left(x_1^2+x_2^2\right)^2}\right) \Rightarrow \sigma_{11} = \frac{\partial^2\Phi}{\partial x_2^2} = -\frac{P}{\pi}\left(\frac{x_1}{\left(x_1^2+x_2^2\right)}+\frac{\left(x_1^2+x_2^2\right)x_1-2x_2^2x_1}{\left(x_1^2+x_2^2\right)^2}\right) = -\frac{2P}{\pi}\frac{x_1^3}{\left(x_1^2+x_2^2\right)^2}$$

$$\frac{\partial\Phi}{\partial x_1} = -\frac{Px_2}{\pi}\left(-\frac{x_2}{\left(x_1^2+x_2^2\right)}\right) \Rightarrow \sigma_{22} = \frac{\partial^2\Phi}{\partial x_1^2} = \frac{Px_2^2}{\pi}\left(\frac{-2x_1}{\left(x_1^2+x_2^2\right)^2}\right) = -\frac{2P}{\pi}\frac{x_1 x_2^2}{\left(x_1^2+x_2^2\right)^2}$$

$$\frac{\partial\Phi}{\partial x_1} = -\frac{Px_2}{\pi}\left(-\frac{x_2}{\left(x_1^2+x_2^2\right)}\right) \Rightarrow \sigma_{12} = \frac{\partial^2\Phi}{\partial x_1 \partial x_2} = \frac{P}{\pi}\left(\frac{-2x_2\left(x_1^2+x_2^2\right)-x_2^2 2x_2}{\left(x_1^2+x_2^2\right)^2}\right) = -\frac{2P}{\pi}\frac{x_1^2 x_2}{\left(x_1^2+x_2^2\right)^2}$$

2. At a given distance from the free surface, along the vertical axis, the following force balance should hold far per unit thickness of the body at a distance $x_1 = a$,

$$-P = \int_{-\infty}^{+\infty} \sigma_{11} dx_2 = -\frac{2P}{\pi}\int_{-\infty}^{+\infty} \frac{a^3}{\left(a^2+x_2^2\right)^2} dx_2$$

Indeed, the integral is equal to $\pi / 2$ and thus, the equilibrium is verified.

Problem 1: Solution

Note that the problem can be solved using cylindrical or Cartesian coordinates. Below both are given.

I. Use cylindrical coordinates:

$x_1 = r\cos\theta; \quad x_2 = r\sin\theta; \quad \tan^{-1} x_2 / x_1 = \theta$

The stress function becomes,

$$\varphi(r,\theta) = cr^2\left[\theta - \frac{1}{2}\sin 2\theta\right] \qquad \text{(a)}$$

With the following derivatives:

$$\begin{aligned}
\frac{\partial\varphi}{\partial r} &= 2cr\left[\theta - \frac{1}{2}\sin 2\theta\right] \\
\frac{\partial^2\varphi}{\partial r^2} &= 2c\left[\theta - \frac{1}{2}\sin 2\theta\right] \\
\frac{\partial\varphi}{\partial\theta} &= cr^2\left[1-\cos 2\theta\right] \\
\frac{\partial^2\varphi}{\partial\theta^2} &= 2cr^2\sin 2\theta
\end{aligned} \qquad \text{(b)}$$

$$\nabla^2\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\varphi = 2c\left[\theta - \frac{1}{2}\sin 2\theta\right] + 2c\left[\theta - \frac{1}{2}\sin 2\theta\right] + 2c\sin 2\theta = 4c\theta$$

$$\nabla^4\varphi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)(4c\theta) = 0+0+0=0$$

The equation (a) can provide the solution.

Stress components:

Use the derivatives in (b)

$$\sigma_{rr} = \frac{1}{r}\frac{\partial\varphi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\varphi}{\partial\theta^2} = 2c\left[\theta - \frac{1}{2}\sin 2\theta\right] + 2c\sin 2\theta = c\left[2\theta + \sin 2\theta\right]$$

$$\sigma_{\theta\theta} = \frac{\partial^2\varphi}{\partial r^2} = 2c\left[\theta - \frac{1}{2}\sin 2\theta\right]$$

$$\sigma_{r\theta} = \frac{1}{r^2}\frac{\partial\varphi}{\partial\theta} - \frac{1}{r}\frac{\partial\varphi}{\partial\theta\partial r} = c\left[1-\cos 2\theta\right] - 2c\left[1-\cos 2\theta\right] = c\left[\cos 2\theta - 1\right]$$

Boundary Conditions:

For $\theta = 0$ $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{r\theta} = 0$ OK

For $\theta = \pi$

$$\sigma_{rr} = c\left[2\theta + \sin 2\theta\right] = c\left[2\pi + \sin 2\pi\right] = 2c\pi; \quad \sigma_{\theta\theta} = 2c\left[\theta - \frac{1}{2}\sin 2\theta\right] = 2c\left[\pi - \frac{1}{2}\sin 2\pi\right] = 2c\pi$$

$$\sigma_{r\theta} = c\left[\cos 2\theta - 1\right] = 0$$

To calculate the constant *c*: for $\theta = \pi \Rightarrow \sigma_{\theta\theta} = 2c\pi = P \Rightarrow c = P/2\pi$

II. Use Cartesian coordinates:

$$\Phi(x_1, x_2) = c\left[\left(x_1^2 + x_2^2\right)\tan^{-1}\frac{x_2}{x_1} - x_1 x_2\right]$$

Derivatives:

$$\frac{\partial\Phi}{\partial x_1} = 2x_1 \tan^{-1}\frac{x_2}{x_1} - 2x_2 \qquad \frac{\partial\Phi}{\partial x_2} = 2x_2 \tan^{-1}\frac{x_2}{x_1}$$

$$\frac{\partial^2\Phi}{\partial x_1^2} = 2\tan^{-1}\frac{x_2}{x_1} - \frac{2x_1x_2}{\left(x_1^2 + x_2^2\right)} \qquad \frac{\partial^2\Phi}{\partial x_2^2} = 2\tan^{-1}\frac{x_2}{x_1} + \frac{2x_1x_2}{\left(x_1^2 + x_2^2\right)}$$

$$\frac{\partial^3\Phi}{\partial x_1^3} = -\frac{4x_2}{\left(x_1^2 + x_2^2\right)} + \frac{4x_1^2x_2}{\left(x_1^2 + x_2^2\right)^2} \qquad \frac{\partial^3\Phi}{\partial x_2^3} = \frac{4x_1}{\left(x_1^2 + x_2^2\right)} - \frac{4x_2^2x_1}{\left(x_1^2 + x_2^2\right)^2}$$

$$\frac{\partial^4\Phi}{\partial x_1^4} = \frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} - \frac{16x_1^3x_2}{\left(x_1^2 + x_2^2\right)^3} \qquad \frac{\partial^4\Phi}{\partial x_2^4} = -\frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{16x_2^3x_1}{\left(x_1^2 + x_2^2\right)^3}$$

$$\frac{\partial^2\Phi}{\partial x_2^2} = 2\tan^{-1}\frac{x_2}{x_1} + \frac{2x_1x_2}{\left(x_1^2 + x_2^2\right)}$$

$$\frac{\partial^3\Phi}{\partial x_2^2\partial x_1} = -\frac{4x_1^2x_2}{\left(x_1^2 + x_2^2\right)^2}$$

$$\frac{\partial^4\Phi}{\partial x_2^2\partial x_1^2} = -\frac{8x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{16x_1^3x_2}{\left(x_1^2 + x_2^2\right)^3}$$

$$\frac{\partial^4\Phi}{\partial x_1^4} + \frac{\partial^4\Phi}{\partial x_2^4} + 2\frac{\partial^4\Phi}{\partial x_2^2\partial x_1^2} = \frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} - \frac{16x_1^3x_2}{\left(x_1^2 + x_2^2\right)^3} - \frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{16x_2^3x_1}{\left(x_1^2 + x_2^2\right)^3} - \frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{32x_1^3x_2}{\left(x_1^2 + x_2^2\right)^3}$$

$$= -\frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} - \frac{16x_1^3x_2}{\left(x_1^2 + x_2^2\right)^3} + \frac{32x_1^3x_2}{\left(x_1^2 + x_2^2\right)^3} + \frac{16x_2^3x_1}{\left(x_1^2 + x_2^2\right)^3}$$

$$= -\frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{-16x_1^3x_2 + 32x_1^3x_2 + 16x_2^3x_1}{\left(x_1^2 + x_2^2\right)^3}$$

$$= -\frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{16x_1^3x_2 + 16x_2^3x_1}{\left(x_1^2 + x_2^2\right)^3} = -\frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{16x_1 x_2 (x_1^2 + x_2^2)}{\left(x_1^2 + x_2^2\right)^3} = -\frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} + \frac{16x_1 x_2}{\left(x_1^2 + x_2^2\right)^2} = 0$$

Stresses:

$$\sigma_{11} = \frac{\partial^2 \Phi}{\partial x_2^2} = 2c\left(\tan^{-1}\frac{x_2}{x_1} + \frac{x_1 x_2}{\left(x_1^2 + x_2^2\right)} \right)$$

$$\sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} = 2c\left(\tan^{-1}\frac{x_2}{x_1} - \frac{x_1 x_2}{\left(x_1^2 + x_2^2\right)} \right)$$

$$\sigma_{12} = -\frac{\partial^2 \Phi}{\partial x_1 \partial x_2} = -c\frac{\partial}{\partial x_2}\left(2x_1 \tan^{-1}\frac{x_2}{x_1} - 2x_2 \right) = 2c\left(1 - \frac{x_1^2}{\left(x_1^2 + x_2^2\right)} \right)$$

Boundary conditions:

$x_2 = 0, \forall x_1 \Rightarrow \sigma_{11} = 0$, $\quad x_2 = 0, x_1 > 0 \Rightarrow \sigma_{22} = 0$, $\quad x_2 = 0, \forall x_1 \Rightarrow \sigma_{12} = 0$

To calculate c. Consider the stress along $x_2 = 0,\ x_1 < 0$

$$\sigma_{22} = \frac{\partial^2 \Phi}{\partial x_1^2} = 2c\left(\tan^{-1}\frac{x_2}{x_1} - \frac{x_1 x_2}{\left(x_1^2 + x_2^2\right)} \right)$$

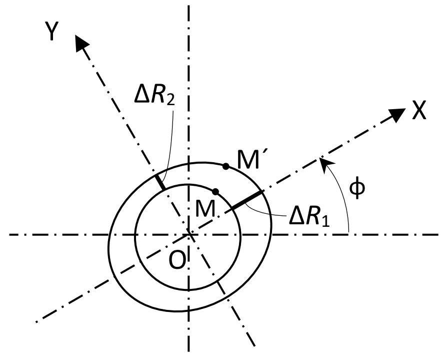
This stress is the only non-zero component. With $x_2 = 0$ the second part in the parentheses is zero and $\tan^{-1}\frac{x_2}{x_1} = 0, \pi ...$ We consider the first non-zero value and equilibrium gives,

$\rightarrow \sigma_{22} = 2c\left(\pi\right) = P \Rightarrow c = P / 2\pi$.

Problem 2: Solution

The stress state is that of plane stress. Consider OXY as the principal system on which we define,

Principal stresses and Their orientation



If the two principal stress σ_1 et σ_2 are positive, the circle deforms as shown in the Figure. A point $M(X,Y)$ on the circle moves to $M'(X',Y')$ defined by,

$$\begin{cases} X' = X + \Delta X = (1+\varepsilon_1)X & \text{avec} \quad \varepsilon_1 = \dfrac{1}{E}(\sigma_1 - \mu\sigma_2) \\ Y' = Y + \Delta Y = (1+\varepsilon_2)Y & \text{avec} \quad \varepsilon_2 = \dfrac{1}{E}(\sigma_2 - \mu\sigma_1) \end{cases}$$

From which $$X^2 + Y^2 = \left(\frac{X'}{1+\varepsilon_1}\right)^2 + \left(\frac{Y'}{1+\varepsilon_2}\right)^2 = R^2 \quad \text{ou} \quad \frac{X'^2}{A^2} + \frac{Y'^2}{B^2} = 1$$

Points M' form an ellipse with semi-axes,

$A = (1+\varepsilon_1)R$ et $B = (1+\varepsilon_2)R$

$\Delta R_1 = A - R = \varepsilon_1 R$ $\Delta R_2 = B - R = \varepsilon_2 R$

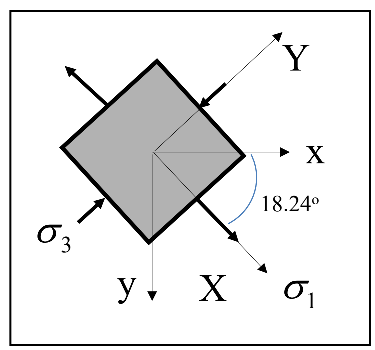
Principal stresses:

$$\sigma_1 = \frac{\sigma_x + \sigma_x}{2} + \sqrt{(\frac{\sigma_x - \sigma_x}{2})^2 + \tau^2} = 100 + 250 = 350\text{MPa}$$

$$\sigma_2 = 0$$

$$\sigma_3 = \frac{\sigma_x + \sigma_x}{2} - \sqrt{(\frac{\sigma_x - \sigma_x}{2})^2 + \tau^2} = 100 - 250 = -150\text{MPa}$$

$$tg 2\varphi_0 = \frac{\tau_x}{\frac{1}{2}(\sigma_x - \sigma_y)} \quad \Rightarrow \quad \varphi_0 = 18°26'$$



Deformation: Using these stresses the strains and the changes in radii are,

$\sigma_1 = 350$ MPa and $\sigma_2 = -150$ MPa $\varepsilon_1 = 1{,}88 \cdot 10^{-3}$ and $\varepsilon_2 = -1{,}21 \cdot 10^{-3}$

$\Delta R_1 = 0{,}0188\text{cm}$; $\Delta R_2 = -0{,}0121\text{cm}$